International Journal of Theoretical Physics, Vol. 36, No. 12, 1997

Finitely Generated Free Modular Ortholattices. II

M. Haviar,[1] P. Konôpka,[1] and C. B. Wegener[2]

Received June 26, 1997

A description is given of the n-generated free algebras $F_{\mathcal{MO}_k}(n)$, $n > 2$, in the varieties $\mathcal{MO}_k$, $k > 2$, of modular ortholattices generated by the ortholattices $\mathbf{MO}_k$ of height 2 with $2k$ atoms. Algebraic methods of the theory of orthomodular lattices are combined with natural duality theory for varieties of algebras. The procedures involved in the analysis of $F_{\mathcal{MO}_k}(n)$ generalize the techniques applied in the preceding paper, where the cases $k = 2$, $n > 2$ were solved. The free algebras are decomposed by central elements into products of canonical intervals. Previous methods are refined to accommodate the fact that the decompositions of $F_{\mathcal{MO}_k}(n)$ lead to intervals of $k - 1$ different types. Their structures are obtained from natural dualities for the varieties $\mathcal{MO}_k$, $k > 2$. Finally, Stirling numbers of the second kind are used to count the number of intervals.The structures of the free algebras $F_{\mathcal{MO}_k}(n)$ for k, $n \leqslant 10$ are explicitly displayed in a table.

1. INTRODUCTION

This is a continuation of Haviar *et al.* (1997), where a more detailed introduction to the topic is provided. We repeat only a few important definitions and results here. The following basic facts about orthomodular lattices can be found in Kalmbach (1983) and Beran (1984).

An *orthomodular lattice* is an algebra $(L; \vee, \wedge, ', 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is the unary operation of orthocomplementation. The operation $'$ is order-reversing with respect to the underlying lattice order $\leqslant$ and the following identities are satisfied:

$$(a')' = a \tag{1}$$

$$a \wedge a' = 0 \qquad \text{and} \qquad a \vee a' = 1 \tag{2}$$

[1]Department of Mathematics, M. Bel University, 974 01 Banská Bystrica, Slovakia; e-mail: haviar@bb.sanet.sk, konopka@fhpv.umb.sk.
[2]Mathematical Institute, University of Oxford, Oxford OX1 3LB, U.K.; e-mail: wegener@maths.ox.ac.uk.

$$(a \wedge b)' = a' \vee b' \qquad \text{and} \qquad (a \vee b)' = a' \wedge b' \tag{3}$$

$$0' = 1, \qquad 1' = 0 \tag{4}$$

$$b = (b \wedge a) \vee [b \wedge (b \wedge a)'] \tag{5}$$

Here (5) is the *orthomodular law.* It has the equivalent form

$$a \leq b \Rightarrow b = a \vee (b \wedge a')$$

Let L be an orthomodular lattice. The *commutator* of elements $x_1, \dots, x_n \in L$ is defined by

$$c(x_1, \dots, x_n) = \bigvee_{(i_1, \dots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n} \tag{6}$$

where $x_i^0 = x_i$ and $x_i^1 = x_i'$. The element $(c(x_1, \dots, x_n))^1$ will be denoted by $c'(x_1, \dots, x_n)$. In particular, the commutator of two elements x, y, which plays an important role in our considerations, is given by

$$c(x, y) = (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

A binary *compatibility relation* $a \leftrightarrow b$ on L is defined by

$$a \leftrightarrow b \qquad \text{if} \quad a = (a \wedge b) \vee (a \wedge b') \qquad (a, b \in L)$$

and satisfies the following rules:

$$a \leq b \Rightarrow a \leftrightarrow b \tag{7}$$

$$a \leq b' \Rightarrow a \leftrightarrow b \tag{8}$$

$$a \leftrightarrow b \Rightarrow a \leftrightarrow b', \quad a' \leftrightarrow b, \quad a' \leftrightarrow b' \tag{9}$$

$$a \leftrightarrow b \Leftrightarrow c(a, b) = 1 \tag{10}$$

In orthomodular lattices the compatibility relation is symmetric and the following version of distributivity related to $\leftrightarrow$ holds: if $M \subseteq L$ is such that $\vee M$ exists in L and $a \in L$ is such that $a \leftrightarrow m$ for every $m \in M$, then

$$a \leftrightarrow \vee M \qquad \text{and} \qquad a \wedge (\vee M) = \bigvee_{m \in M} (a \wedge m) \tag{11}$$

Using the previous rules, it is easy to show that

$$c(x_1, \dots, x_n) \leftrightarrow x_i \qquad \text{for every} \quad i = 1, 2, \dots, n \tag{12}$$

and

$$c(x_1, \dots, x_n) \leftrightarrow t(x_1, \dots, x_n) \tag{13}$$

for any n-ary term t and any $x_1, \dots, x_n \in L$.

We shall often deal with intervals in L of the form $[0, v]$ ($v \in L$). These intervals can be considered as orthomodular lattices if one defines the orthocomplement of an element $a \in [0, v]$ in $[0, v]$ to be $a' \wedge v$, where a' is the complement in L.

Elements $a \in L$ which are compatible with every $x \in L$ are called *central.* The set $Z(L)$ of all central elements of L is a Boolean subalgebra of L, called the *center* of L. Moreover,

$$a \in Z(L), v \in L \Rightarrow a \wedge v \in Z([0, v]) \qquad (14)$$

The following fact about orthomodular lattices (Kalmbach, 1983, p. 20) enables us to decompose the free algebras in question into products of smaller lattices, the structures of which are more readily analyzed:

$$c \in Z(L) \Leftrightarrow L \cong [0, c] \times [0, c'] \qquad (15)$$

The finitely generated free algebras under consideration lie within certain subvarieties of the variety of all orthomodular lattices. Let us recall some basic facts about the subvariety lattice of the variety of all orthomodular lattices $\mathcal{OM}$ (Kalmbach, 1983, Chapter 2.9). At the bottom there is a three-element covering chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2$$

(see Fig. 1), where $\mathcal{T}$ and $\mathcal{B}$ are the varieties of trivial algebras and Boolean algebras, respectively, and $\mathcal{MO}_2 = \mathbf{V}(\mathbf{MO}_2)$ is the variety generated by the orthomodular lattice $\mathbf{MO}_2$ of height 2 with 4 atoms a, a', b, b' (see Fig. 2).

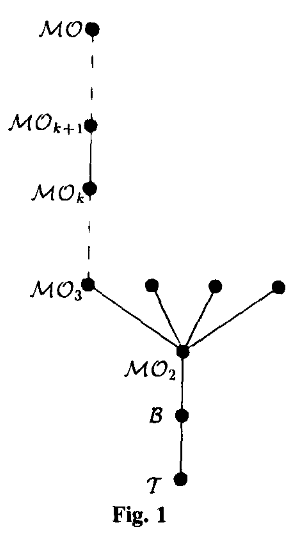

Fig. 1

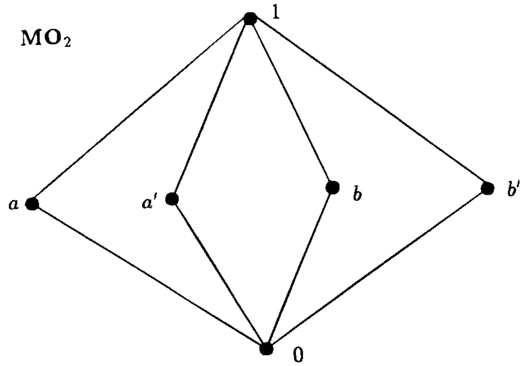

Fig. 2

In general, $\mathbf{MO}_k$ ($k \geq 2$) denotes the orthomodular lattice of height 2 with $2k$ atoms. A *block* of $\mathbf{MO}_k$ is a maximal Boolean subalgebra $\{0, a, a', 1\}$ of $\mathbf{MO}_k$ where a is an atom. It is easy to see that each $\mathbf{MO}_k$ ($k \geq 2$) satisfies the modular law

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$$

Let $\mathcal{MO}$ denote the variety of all modular ortholattices. The only finite subdirectly irreducible algebras in this variety are $\mathbf{MO}_k$ ($k \geq 2$) and **2.** Hence the subvarieties of $\mathcal{MO}$ form the chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2 \subsetneq \mathcal{MO}_3 \subsetneq \cdots \subsetneq \mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1} \subsetneq \cdots \subsetneq \mathcal{MO}$$

of type $\omega + 1$, where $\mathcal{MO}_k = \mathbf{V}(\mathbf{MO}_k)$ is the variety generated by $\mathbf{MO}_k$. The strict inclusions $\mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1}$ are given by the fact that the generator $\mathbf{MO}_k$ satisfies

$$\bigwedge_{\substack{i,j=1 \\ i<j}}^{k+1} c'(x_i, x_j) = 0$$

but $\mathbf{MO}_{k+1}$ does not.

We use the notation $F_{\mathbf{V}}(n)$ for the free algebra with n generators in a variety **V.** Clearly,

$$F_{\mathcal{OM}}(1) = F_{\mathcal{B}}(1) \cong \mathbf{2}^2$$

where **2** denotes the two-element Boolean algebra $\mathbf{2} = (\{0, 1\}; \vee, \wedge, ', 0, 1)$. Further (Beran, 1984, III.2),

$$F_{\mathcal{OM}}(2) \cong F_{\mathcal{B}}(2) \times \mathbf{MO}_2 \cong \mathbf{2}^4 \times \mathbf{MO}_2 \cong F_{\mathcal{MO}_2}(2)$$

The free algebra $F_{\mathcal{MO}}(3)$ is infinite since it has the orthomodular lattice of closed subspaces of $\mathbb{R}^3$ as a homomorphic image. However, the algebras

$F_{\mathcal{MO}_k}(n)$ ($k \geq 2$, $n \geq 3$) are finite because the varieties $\mathcal{MO}_k$ are locally finite (Clark and Davey, 1998, Chapter 1.3).

In the preceding paper we described the free algebras $F_{\mathcal{MO}_2}(n)$ in the variety $\mathcal{MO}_2$ for every $n > 2$ and determined their cardinalities. Let us recall the main results.

Theorem 1.1 (Haviar, *et al.*, 1997, Theorem 3.3). For any $n \geq 1$,

$$F_{\mathcal{MO}_2}(n) \cong F_{\mathcal{B}}(n) \times (\mathbf{MO}_2)^{\phi(n)}$$

where

$$\phi(n) = 2^{n-3} \cdot (3^n - 2^{n+1} + 1)$$

Corollary 1.2 (Haviar *et al.*, 1997, Corollary 3.4). For any $n \geq 1$,

$$|F_{\mathcal{MO}_2}(n)| = 2^{2^n} \cdot 6^{2^{n-3} \cdot (3^n - 2^{n+1} + 1)}$$

To obtain these results, we first decomposed $F_{\mathcal{MO}_2}(n)$ by suitable central elements into simple canonical intervals of the form $[0, C_G]$ for certain term functions $C_G = C_G(x_1, \ldots, x_n)$. Then we showed that each such interval $[0, C_G]$ was isomorphic to $(\mathbf{MO}_2)^{2n-2}$ by using methods of natural duality theory (Davey and Werner, 1983; Clark and Davey, 1998). This was the crucial part of our method. Finally, we used ordinary combinatories to count the number of canonical intervals $[0, C_G]$ by establishing a one-to-one correspondence between the term functions $C_G(x_1, \ldots, x_n)$ and n-element graphs G containing a complete bipartite subgraph and isolated vertices. This technique was illustrated by a detailed discussion of the case $F_{\mathcal{MO}_2}(3)$ in Haviar *et al.* (1997).

The aim of this paper is to generalize this method to describe finitely generated free modular ortholattices $F_{\mathcal{MO}_k}(n)$ with n generators in the varieties $\mathcal{MO}_k$, where $n > 2$, $k > 2$. The situation is slightly more complex here. As it turns out, the decomposition of the free algebra $F_{\mathcal{MO}_k}(n)$ ($k > 2$, $n > 2$) consists of canonical intervals $[0, C_{G_p}(x_1, \ldots, x_n)]$ of $k - 1$ different types corresponding to graphs G_p, $p \in \{2, \ldots, k\}$, containing a complete p-partite subgraph and isolated vertices. To describe the canonical intervals $[0, C_{G_p}]$ in $F_{\mathcal{MO}_k}(n)$, we need to build up an effective natural duality for the variety $\mathcal{MO}_k$. To count the number of graphs of each type, we use the Stirling numbers of the second kind. We present formulas for the structure of $F_{\mathcal{MO}_k}(n)$ and its cardinality, and give a table showing explicitly the structures of $F_{\mathcal{MO}_k}(n)$ for $k, n \leq 10$.

2. NATURAL DUALITIES FOR THE VARIETIES $\mathcal{MO}_k$

For the basic facts about natural duality theory we recommend Davey (1993) and Clark and Davey (1998). A very brief summary of the basic concepts can be found in Section 2 of Haviar, *et al.* (1997). Here we recall only a few facts of the theory.

Let $\underline{\mathbf{M}} = (M; F)$ be a finite algebra. Let $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ be the discrete topological structure, in which the set M is endowed with the discrete topology τ and with (finite) families G, H, and R of operations, partial operations, and relations, respectively. By a *graph* of an n-ary (partial) operation $h: M^n \to M$ we mean an $(n + 1)$-ary relation

$$h^\square = \{(x_1, \ldots, x_n, h(x)) | (x_1, \ldots, x_n) \in M^n\} \subseteq M^{n+1}$$

The structure $\underset{\sim}{\mathbf{M}}$ is said to be *algebraic over* $\underline{\mathbf{M}}$ if the relations in R and the graphs of operations and partial operations in $G \cup H$ are subalgebras of appropriate powers of $\underline{\mathbf{M}}$. Throughout this paper we are assuming that $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ is algebraic over $\underline{\mathbf{M}}$.

Let $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$ be the quasi-variety generated by $\underline{\mathbf{M}}$ and let $\mathcal{X} = \mathrm{IS_CP}(\underset{\sim}{\mathbf{M}})$ be the class of all structures which are embeddable as closed substructures into powers of $\underset{\sim}{\mathbf{M}}$. Let $\mathbf{A}$ be an algebra in $\mathcal{A}$ and let $D(\mathbf{A})$ denote the set of all $\mathcal{A}$-homomorphisms $\mathbf{A} \to \underline{\mathbf{M}}$. Similarly, for $\mathbf{X} \in \mathcal{X}$, let $E(\mathbf{X})$ denote the set of all $\mathcal{X}$-morphisms $\mathbf{X} \to \underset{\sim}{\mathbf{M}}$, which are the continuous maps preserving the graphs of all (partial) operations in $G \cup H$ and all relations in R. (We recall that for any set $X \subseteq M^I$, a map $\varphi: X \to M$ preserves the relation $r \subseteq M^n$ if, whenever $\tilde{x}_1 = (x_{1i})_{i \in I}, \ldots, \tilde{x}_n = (x_{ni})_{i \in I}$ are such that $[x_{1i}, \ldots, x_{ni}] \in r$ for every $i \in I$, then $[\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_n)] \in r$.) Since $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, $D(\mathbf{A})$ and $E(\mathbf{X})$ can be understood as members of $\mathcal{X}$ and $\mathcal{A}$, respectively, where we endow these sets of maps pointwise with the structures of $\underset{\sim}{\mathbf{M}}$ and $\underline{\mathbf{M}}$.

Let $\mathbf{A} \in \mathcal{A}$, $\mathbf{X} \in \mathcal{X}$ and let e_A: $\mathbf{A} \to ED(\mathbf{A})$ and ϵ_X: $\mathbf{X} \to DE(\mathbf{X})$ be given by evaluation:

$$e_A(a)(h) = h(a) \qquad \text{for every} \quad a \in A \quad \text{and} \quad h \in D(\mathbf{A})$$

$$\epsilon_X(y)(\varphi) = \varphi(y) \qquad \text{for every} \quad y \in X \quad \text{and} \quad \varphi \in E(\mathbf{X})$$

The maps e_A, ϵ_X are embeddings whenever $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, in which case we say that $\underset{\sim}{\mathbf{M}}$ *yields a pre-duality on* $\mathcal{A}$. The structure $\underset{\sim}{\mathbf{M}}$ (or just $G \cup H \cup R$) is said to *yield a (natural) duality on* $\mathcal{A}$ if for every $\mathbf{A} \in \mathcal{A}$ the embedding e_A is an isomorphism. In this case every algebra $\mathbf{A}$ in $\mathcal{A}$ is isomorphic to the algebra $ED(\mathbf{A})$ of all continuous $(G \cup H \cup R)$-preserving maps from $D(\mathbf{A})$ to $\underline{\mathbf{M}}$, a representation which allows us to formulate Theorem 2.2. We say that $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$) *entails* an n-ary (partial) operation h if for every $\mathbf{X} \in \mathcal{X}$, each member of $E(\mathbf{X})$ preserves the graph $h^\square$ as an $(n + 1)$-ary relation. The structure $\underset{\sim}{\mathbf{M}}$ entails a set of (partial) operations K if it entails each $k \in K$. The role which entailment plays in order to obtain workable dualities is discussed in Davey. *et al.* (1995). Let us quote two results here. If e and r are unary algebraic (partial) operations on M [i.e., (partial) endomorphisms of $\underline{\mathbf{M}}$], then $\{e, r\}$ entails the composition $r \circ e$ and the intersection $r^{\square} \cap e^{\square}$.

A variety generated by an algebra $\underline{\mathbf{M}}$ is arithmetical if and only if $\underline{\mathbf{M}}$ has an arithmeticity (Pixley) term function $p(x, y, z)$: $\underline{\mathbf{M}}^3 \to \underline{\mathbf{M}}$ satisfying

$$p(a, b, b) = p(a, b, a) = p(b, b, a) = a \qquad \text{for all} \quad a, b \in M$$

Now we are ready to repeat a result from Haviar *et al.* (1997, Theorem 2.1) which is an immediate consequence of the Unary Partial Algebra Theorem in Clark and Davey (1996).

Theorem 2.1. Assume that a subdirectly irreducible algebra $\underline{\mathbf{M}}$ generates an arithmetical variety $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$. Let $\mathcal{P}_1$ be the set of all unary (partial) endomorphisms of $\underline{\mathbf{M}}$. Then any set H of unary (partial) endomorphisms of $\underline{\mathbf{M}}$ that entails $\mathcal{P}_1$ yields a duality on $\mathcal{A}$.

It is well known that an n-generated free algebra in the variety generated by $\underline{\mathbf{M}}$ is isomorphic to the algebra of all n-ary term functions on $\underline{\mathbf{M}}$. A further description of this algebra is obtained by natural duality theory (see also Haviar *et al.*, 1997, Theorem 2.2).

Theorem 2.2. Let $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ be a variety and let $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ yield a duality on $\mathcal{A}$. The n-generated free algebra $F_{\mathcal{A}}(n)$ in the variety generated by $\underline{\mathbf{M}}$ is isomorphic to the algebra of all $(G \cup H \cup R)$-preserving functions from $\mathbf{M}^n$ to $\mathbf{M}$.

If $G \cup H \cup R$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{M}})$, then the two representations of the free algebras are the same, as it can be shown that the continuous $(G \cup H \cup R)$-preserving functions from $\mathbf{M}^n$ to $\mathbf{M}$ are exactly the n-ary term functions on $\underline{\mathbf{M}}$ (Davey, 1993, p. 87).

We may now turn our attention to the varieties $\mathcal{MO}_k = \mathbf{V}(\mathbf{MO}_k)$ which coincide with the quasi-varieties $\mathbb{ISP}$ $(\mathbf{MO}_k)$ (Clark and Davey, 1998, Chapter 1.3). In order to use the representation of finitely generated algebras in $\mathcal{MO}_k$ in terms of $(G \cup H \cup R)$-preserving maps, we need to exhibit a workable duality for the varieties $\mathcal{MO}_k$. The term function

$$\begin{aligned} p(x, y, z) = {} & (x \vee z) \wedge (x \vee y') \wedge (z \vee y') \\ & \wedge [(c(x, y) \wedge z) \vee (c(y, z) \wedge x) \vee (c(x, z) \wedge x \wedge z)] \end{aligned}$$

is an arithmeticity term function for the generator $\mathbf{MO}_k$, because if x, z belong to the same block of $\mathbf{MO}_k$, then $(x \vee z) \wedge (x' \vee z) = z$ and $c(x, z) = 1$; and if x, z are atoms of different blocks of $\mathbf{MO}_k$, $(x \vee z) \wedge (x' \vee z) = 1$ and $c(x, z) = 0$. Thus, by Theorem 2.1 any set H which entails the set of all unary (partial) endomorphisms of $\mathbf{MO}_k$, $\mathcal{P}_1$, yields a duality on $\mathcal{MO}_k$. For

$k \geq 2$, every (total) endomorphism of $\mathbf{MO}_k$ is an automorphism. Each partial endomorphism of $\mathbf{MO}_k$ must map the top to the top and the bottom to the bottom. If a partial endomorphism maps an atom a to $c \in \{0, 1\}$, then it must map a' to $c' \in \{0, 1\}$, and such partial endomorphisms are not extendable. Any other partial endomorphism must map all atoms in its domain to distinct atoms of $\mathbf{MO}_k$, while preserving the complementation. Partial endomorphisms of this kind extend to automorphisms, and their graphs can be obtained by intersection from the automorphism group, Aut($\mathbf{MO}_k$). Similarly the partial endomorphism with graph $\{(0, 0), (1, 1)\}$ is entailed by Aut($\mathbf{MO}_k$). So let us consider a nonextendable partial endomorphism r mapping onto $\{0, 1\}$, with graph $r^\square = \{(0, 0), (a, 0), (a', 1), (1, 1)\}$, where a is some atom in $\mathbf{MO}_k$. Then $H = \text{Aut}(\mathbf{MO}_k) \cup \{r\}$ entails all partial endomorphisms of the same type as r by composition of the automorphisms with r. Since intersection and composition of partial endomorphisms are admissible entailment constructs, $H = \text{Aut}(\mathbf{MO}_k) \cup \{r\}$ entails $\mathcal{P}_1$ and hence the next result is an immediate consequence of Theorem 2.1.

Theorem 2.3. Let a be an atom of $\mathbf{MO}_k$ and let r be the partial endomorphism with graph $r^\square = \{(0, 0), (a, 0), (a', 1), (1, 1)\}$. Then for $k \geq 2$, $H = \text{Aut}(\mathbf{MO}_k) \cup \{r\}$ yields a duality on the variety $\mathcal{MO}_k = \mathbb{ISP}(\mathbf{MO}_k)$.

We may formulate a corollary to Theorems 2.2 and 2.3.

Corollary 2.4. Let $H = \text{Aut}(\mathbf{MO}_k) \cup \{\text{r}\}$. Then the n-generated free algebra $F_{\mathcal{MO}_k}(n)$ in the variety $\mathcal{MO}_k$ is isomorphic to the algebra of all H-preserving functions from $(\mathbf{MO}_k)^n$ to $\mathbf{MO}_k$.

3. FINITELY GENERATED FREE ALGEBRAS IN $\mathcal{MO}_k$

Let $F_{\mathcal{MO}_k}(n)$ denote the free orthomodular lattice on n generators in the variety $\mathcal{MO}_k = \mathbb{ISP}(\mathbf{MO}_k)$. In the last section we showed that $F_{\mathcal{MO}_k}(n)$ is isomorphic to the algebra of all those functions from $(\mathbf{MO}_k)^n$ to $\mathbf{MO}_k$ which preserve $H = \text{Aut}(\mathbf{MO}_k) \cup \{r\}$ and noted that these functions are exactly the n-ary term functions on $\mathbf{MO}_k$. This representation allows us to find central elements to decompose $F_{\mathcal{MO}_k}(n)$ into a product of intervals which we can evaluate in terms of H-preserving functions.

The first stage in the analysis of the structure of $F_{\mathcal{MO}_k}(n)$ is to find central elements within $F_{\mathcal{MO}_k}(n)$ by which to decompose $F_{\mathcal{MO}_k}(n)$. Let $t(x_1, \ldots, x_n)$: $(\mathbf{MO}_k)^n \to \mathbf{MO}_k$ be a term function into $\{0, 1\}$. Then for any term function $u(x_1, \ldots, x_n)$: $(\mathbf{MO}_k)^n \to \mathbf{MO}_k$,

$$t(x_1, \ldots, x_n) = (t(x_1, \ldots, x_n) \wedge u(x_1, \ldots, x_n)) \vee (t(x_1, \ldots, x_n) \wedge u'(x_1, \ldots, x_n))$$

and therefore any term function $t(x_1, \ldots, x_n)$ mapping into $\{0, 1\}$ is a central element of $F_{\mathcal{MO}_k}(n)$. Commutators take only values 0 and 1 and hence by (15) we may write

$$F_{\mathcal{MO}_k}(n) = [0, c(x_1, \ldots, x_n)] \times [0, c'(x_1, \ldots, x_n)]$$

Let us analyze the structure of the interval $[0, c(x_1, \ldots, x_n)]$ first.

Theorem 3.1. The interval $[0, c(x_1, \ldots, x_n)]$ in $F_{\mathcal{MO}_k}(n)$ is isomorphic to the n-generated free Boolean algebra $F_{\mathcal{B}}(n)$. Hence

$$[0, c(x_1, \ldots, x_n)] \cong \mathbf{2}^{2^n}$$

Proof. We define n-ary functions a_i: $(\mathbf{MO}_k)^n \to \mathbf{MO}_k$, $i = 1, \ldots, 2^n$, in the following way:

$$a_1 = x_1' \wedge x_2' \wedge x_3' \wedge \cdots \wedge x_n' \wedge c(x_1, \ldots, x_n)$$

$$a_2 = x_1 \wedge x_2' \wedge x_3' \wedge \cdots \wedge x_n' \wedge c(x_1, \ldots, x_n)$$

$$\cdots$$

$$a_{n+1} = x_1' \wedge x_2' \wedge \cdots \wedge x_{n-1}' \wedge x_n \wedge c(x_1, \ldots, x_n)$$

$$a_{n+2} = x_1 \wedge x_2 \wedge x_3' \cdots \wedge x_n' \wedge c(x_1, \ldots, x_n)$$

$$\cdots$$

$$a_{2^n} = x_1 \wedge x_2 \wedge x_3 \cdots \wedge x_n \wedge c(x_1, \ldots, x_n)$$

One can easily show that $a_i \leq a_j'$ (i.e., a_i, a_j are *orthogonal*) for all $i, j \in \{1, \ldots, 2^n\}$, $i \neq j$, hence by (8), $a_i \leftrightarrow a_j$. Now we use the following fact about orthomodular lattices (Pták and Pulmanová, 1991, Propositions 1.3.27 and 1.3.29): if A is a subset of an orthomodular lattice L such that every two elements of A are compatible, then A can be embedded into a Boolean subalgebra of L. Let us put $A = \{a_1, \ldots, a_{2^n}\} \subset [0, c(x_1, \ldots, x_n)]$. Then obviously A generates a Boolean subalgebra of $[0, c(x_1, \ldots, x_n)]$, say B with 2^n atoms $a_1, \ldots, a_{2^n}$, which is isomorphic to the free Boolean algebra $\mathbf{2}^{2^n}$. It remains to show that every element of the interval $[0, c(x_1, \ldots, x_n)]$ belongs to B.

Note that by (6)–(8) and (11)

$$\begin{aligned} x_i \wedge c(x_1, \ldots, x_n) &= x_i \wedge \left(\bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n} \right) \\ &= \bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} (x_i \wedge x_1^{i_1} \wedge \cdots \wedge x_n^{i_n}) \\ &= \bigvee (a_j \in A | a_j \leq x_i) \end{aligned} \tag{16}$$

$$\begin{aligned} x_i' \wedge c(x_1, \ldots, x_n) &= x_i' \wedge \left(\bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n} \right) \\ &= \bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} (x_i' \wedge x_1^{i_1} \wedge \cdots \wedge x_n^{i_n}) \\ &= \bigvee (a_j \in A \mid a_j \leq x_i') \end{aligned} \tag{17}$$

for all $i = 1, \ldots, n$. Each element $b \in [0, c(x_1, \ldots, x_n)]$ is of the form $b = t(x_1, \ldots, x_n) \wedge c(x_1, \ldots, x_n)$ for some n-ary term t. Using (3), the term $t(x_1, \ldots, x_n)$ can be written in a form $l(x_1, \ldots, x_n, x_1', \ldots, x_n')$, where $l(z_1, \ldots, z_{2n})$ is a lattice term in which $x_1, \ldots, x_n, x_1', \ldots, x_n'$ are substituted for $z_1, \ldots, z_{2n}$. Since $c = c(x_1, \ldots, x_n)$ is compatible with every element of $F_{\mathcal{MO}_k}(n)$, by (11) we have

$$\begin{aligned} t(x_1, \ldots, x_n) \wedge c &= l(x_1, \ldots, x_n, x_1', \ldots, x_n') \wedge c \\ &= l(x_1 \wedge c, \ldots, x_n \wedge c, x_1' \wedge c, \ldots, x_n' \wedge c) \end{aligned}$$

Using the formulas in (16) and (17) for $x_1 \wedge c, \ldots, x_n \wedge c$ and $x_1' \wedge c, \ldots, x_n' \wedge c$, respectively, b can be expressed as a Boolean term function $b(a_1, \ldots, a_n)$. This shows that b lies in B, completing the proof. ■

To evaluate $[0, c'(x_1, \ldots, x_n)]$ we decompose this interval further. Recall that if a is a central element in the orthomodular lattice L and v is an element of L, then $a \wedge v$ is central in $[0, v] \subseteq L$. Commutators are central elements in $F_{\mathcal{MO}_k}(n)$, hence we may use the commutators $c(x_i, x_j)$ for $i, j = 1, \ldots, n$, where $i < j$, to arrive at the decomposition

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_{\vec{w} \in \{0,1\}^N} \left[0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_l, \ldots, x_n) \right]$$

where the product is taken over all N-tuples

$$\vec{w} = (w_{1,2}, \ldots, w_{1,n}, w_{2,3}, \ldots, w_{n-1,n}) \in \{0, 1\}^N$$

where $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j) & \text{if} \quad w_{i,j} = 0 \\ c'(x_i, x_j) & \text{if} \quad w_{i,j} = 1 \end{cases}$$

As in our previous paper (Haviar *et al.*, 1997) we may construct a labeled unoriented graph $G_{\vec{w}}$ (without multiple edges and loops) for every term function

$$t_{\tilde{w}}(x_1, \ldots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

on a vertex set $\{x_1, \ldots, x_n\}$ with edges $x_i x_j$ whenever $w_{i,j} = 1$ for $i < j$. Given such a graph G, we are able to reconstruct the term function $t_{\tilde{w}}$, also denoted by C_G. Thus any one of $\tilde{w}$, $t_{\tilde{w}}$ $(=C_G)$, and G determines the other two. To analyze the structure of $[0, c'(x_1, \ldots, x_n)]$ we need to evaluate the intervals $[0, t_{\tilde{w}}(x_1, \ldots, x_n)]$ for every N-tuple $\tilde{w}$. Some of these intervals are trivial; Proposition 3.2 gives a necessary and sufficient condition on the structure of the corresponding graph G for $[0, t_{\tilde{w}}(x_1, \ldots, x_n)] = [0, C_G(x_1, \ldots, x_n)]$ to be nontrivial.

Proposition 3.2. Let $C_G(x_1, \ldots, x_n)$: $(\mathbf{MO}_k)^n \to \mathbf{MO}_k$ be the term function

$$\bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

and G be the associated graph. Then the following conditions are equivalent:

(a) $C_G(x_1, \ldots, x_n)$ is not identically equal to zero;

(b) there exist elements $a_1, \ldots, a_n \in \mathbf{MO}_k$ with the following properties:
(i) $C_G(a_1, \ldots, a_n) = 1$;
(ii) the elements $a_1, \ldots, a_n$ are not all from the same block of $\mathbf{MO}_k$;
(iii) $x_i x_j$ is an edge of G if and only if a_i, a_j are atoms of different blocks in $\mathbf{MO}_k$;

(c) $G_p := G$ consists of l isolated vertices $(0 \leq l \leq n - p)$ and one connected component which is a complete p-partite graph $(2 \leq p \leq n)$.

Moreover, provided $G = G_p$ is as in (c), then there are exactly $2^n\binom{k}{p}p!$ n-tuples $(a_1, \ldots, a_n) \in (\mathbf{MO}_k)^n$ such that $C_G(a_1, \ldots, a_n)$ is nonzero.

Proof. (a) $\Rightarrow$ (b). Suppose (a) holds, then there exist $a_1, \ldots, a_n \in \mathbf{MO}_k$ such that $C_G(a_1, \ldots, a_n) \neq 0$. This implies that $c^{w_{i,j}}(a_i, a_j)$ and $c'(a_1, \ldots, a_n)$ are nonzero for all i and j, and hence must be 1, which forces $C_G(a_1, \ldots, a_n) = 1$. Now $c'(a_1, \ldots, a_n)$ is equal to 1 if and only if there exist i and j such that a_i, a_j are atoms of different blocks of $\mathbf{MO}_k$. For such i, j, where $i < j$, $c^{w_{i,j}}(a_i, a_j) = 1$ if and only if $w_{i,j} = 1$ if and only if $x_i x_j$ is an edge in G, proving (b).

(b) $\Rightarrow$ (c). Let $a_1, \ldots, a_n \in \mathbf{MO}_k$ be as in condition (b). By (b)(iii), for $a_i \in \{0, 1\}$, x_i must be an isolated vertex in G. If a_i is an atom in $\mathbf{MO}_k$, then by (b)(ii) there exists j such that a_j is an atom of a different block and by (b)(iii), for all such i,j there is an edge $x_i x_j$ in G. Thus G has isolated vertices associated with those $a_i \in \{a_1, \ldots, a_n\}$ which are 0 or 1 and the other vertices may be partitioned according to which block the corresponding a_i come from, giving a complete p-partite graph by (b)(iii), with p greater than or equal to 2 by (b)(ii). This proves (c).

(c) $\Rightarrow$ (a). We show that, given a labeled graph $G = G_p$ as in (c), we may choose $a_1, \ldots, a_n \in \mathbf{MO}_k$ such that $C_G(a_1, \ldots, a_n)$ is nonzero: the value of C_G at $(a_1, \ldots, a_n)$ is nonzero if and only if all the expressions $c^{w_{i,j}}(a_i, a_j)$ and $c'(a_1, \ldots, a_n)$ are 1. Let us consider the connected component of G first, which is partitioned into $p \geq 2$ parts. Let x_i be a vertex in the connected component. Then for every j such that x_ix_j is an edge of G, C_G contains the term $c'(x_i, x_j)$ if $i < j$ or $c'(x_j, x_i)$ if $j < i$. This term obviously takes value 1 at (a_i, a_j) if and only if we choose a_i, a_j from different blocks of $\mathbf{MO}_k$. For x_j lying in the same block of the p-partite graph as x_i, C_G contains the term $c(x_i, x_j)$ if $i < j$ or $c(x_j, x_i)$ if $j < i$. This term takes value 1 at (a_i, a_j) if and only if a_i, a_j are from the same block of $\mathbf{MO}_k$. If x_i is an isolated vertex in G, then any term $c^{w_{i,j}}(x_i, x_j)$ in C_G becomes $c(x_i, x_j)$ [and similarly for $c^{w_{j,i}}(x_j, x_i)$], so a_i has to lie in the same block as a_j for all j. This forces us to choose a_i to be either 0 or 1. So to make C_G nonzero at $(a_1, \ldots, a_n)$ we allocate a unique block of $\mathbf{MO}_k$ to each block of the p-partite component of G and choose the corresponding a_i to be atoms of the associated blocks. For isolated x_i we choose $a_i \in \{0, 1\}$. This proves (a).

The above discussion allows us to count the number of n-tuples $(a_1, \ldots, a_n)$ at which C_G is nonzero. We saw that we needed to allocate p blocks of $\mathbf{MO}_k$ in any order to the p blocks of the connected p-partite component. There are two choices for any a_i once the order of the blocks has been chosen, namely either of the two atoms in the corresponding block for x_i in the connected component, or 0 or 1 for isolated x_i, giving $2^n\binom{k}{p}p!$ such n-tuples $(a_1, \ldots, a_n)$. ■

The next task is to analyze the structure of the intervals $[0, C_G(x_1, \ldots, x_n)]$ associated with graphs $G = G_p$ described in Proposition 3.2(c). This can be done using the duality for $\mathcal{MO}_k$ given by $H = \mathrm{Aut}(\mathbf{MO}_k) \cup \{r\}$. The interval $[0, C_G(x_1, \ldots, x_n)]$ is isomorphic to the algebra of all those H-preserving functions $f{:}(\mathbf{MO}_k)^n \to \mathbf{MO}_k$ (which are the same as the n-ary term functions on $\mathbf{MO}_k$) which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$. Any such function f must take value zero whenever the term C_G does. Let T_G be the set consisting of the $2^n\binom{k}{p}p!$ n-tuples $(a_1, \ldots, a_n)$ from $(\mathbf{MO}_k)^n$ at which C_G is nonzero, that is $C_G(a_1, \ldots, a_n) = 1$.

The function f: $(\mathbf{MO}_k)^n \to \mathbf{MO}_k$ preserves a (partial) endomorphism e with graph $e^\square$ if for $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$, $\underline{\mathbf{b}} = (b_1, \ldots, b_n) \in (\mathbf{MO}_k)^n$,

$$(a_1, b_1) \in e^\square, \ldots, (a_n, b_n) \in e^\square \Rightarrow (f(\underline{\mathbf{a}}), f(\underline{\mathbf{b}})) \in e^\square \tag{18}$$

Let r be the partial endomorphism with graph $r^{\square} = \{(0, 0), (a, 0), (a', 1), (1, 1)\}$, where a is an atom of $\mathbf{MO}_k$. For the left-hand side of (18) to hold, the elements a_i must lie in $\{0, a, a', 1\}$ and the elements b_i in $\{0, 1\}$, hence neither $\underline{\mathbf{a}}$ nor $\underline{\mathbf{b}}$ can lie in T_G. Therefore $(f(\underline{\mathbf{a}}), f(\underline{\mathbf{b}})) = (0, 0) \in r^{\square}$ for any $f \leq C_G$, making f automatically r-preserving.

Before we discuss the preservation of the automorphisms, let us consider the action of the automorphism group $\mathrm{Aut}(\mathbf{MO}_k)$ on $(\mathbf{MO}_k)^n$. The following concepts and basic facts about group actions can be found, for example, in Neumann *et al.* (1994). The group $\mathrm{Aut}(\mathbf{MO}_k)$ acts in the natural way on $\mathbf{MO}_k$ by permuting the atoms. For an automorphism α and an element a in $\mathbf{MO}_k$ let us denote the action of α on a by a^{α} [we could also write $\alpha(a)$ or $a\alpha$ depending on whether we treat α as a function or a permutation]. We may extend the action of $\mathrm{Aut}(\mathbf{MO}_k)$ on $\mathbf{MO}_k$ pointwise to $(\mathbf{MO}_k)^n$, so for $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in (\mathbf{MO}_k)^n$ and $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$, $a^{\alpha} = (a_1^{\alpha}, \ldots, a_n^{\alpha}) \in (\mathbf{MO}_k)^n$. For such $\underline{\mathbf{a}}$ and α we denote the orbit of $\underline{\mathbf{a}}$ by

$$\mathrm{Orb}\ \underline{\mathbf{a}} = \{\underline{\mathbf{a}}^{\beta} \mid \beta \in \mathrm{Aut}(\mathbf{MO}_k)\}$$

the stabilizer of $\underline{\mathbf{a}}$ by

$$\mathrm{Stab}\ \underline{\mathbf{a}} = \{\beta \in \mathrm{Aut}(\mathbf{MO}_k) \mid \underline{\mathbf{a}}^{\beta} = \underline{\mathbf{a}}\}$$

and the set of elements kept fixed by α under the action on $\mathbf{MO}_k$ by

$$\mathrm{fix}_{\mathbf{MO}_k}\alpha = \{b \in \mathbf{MO}_k \mid b^{\alpha} = b\}$$

A version of Lagrange's theorem (Neumann *et al.*, 1994, Corollary 6.2) asserts that, for all $\underline{\mathbf{a}} \in (\mathbf{MO}_k)^n$,

$$|\mathrm{Aut}(\mathbf{MO}_k)| = |\mathrm{Orb}\ \underline{\mathbf{a}}| \cdot |\mathrm{Stab}\ \underline{\mathbf{a}}| \tag{19}$$

To compute the size of $|\mathrm{Aut}(\mathbf{MO}_k)|$ for $k \geq 2$, note that any automorphism is determined by the images of k atoms, one from each block, which have to be mapped to atoms of distinct blocks of $\mathbf{MO}_k$, giving two choices per such atom once the order of blocks has been fixed. Hence $|\mathrm{Aut}(\mathbf{MO}_k)| = 2^k k!$.

We may rewrite (18) for automorphisms $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$. The function $f{:}(\mathbf{MO}_k)^n \to \mathbf{MO}_k$ is α-preserving if for all $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in (\mathbf{MO}_k)^n$,

$$f(\underline{\mathbf{a}}^{\alpha}) = f(\underline{\mathbf{a}})^{\alpha} \tag{20}$$

Let us return to the interval $[0, C_G(x_1, \ldots, x_n)]$ associated with a graph $G = G_p$ $(2 \leq p \leq k)$. For any $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$,

$$\underline{\mathbf{a}} \in T_G \quad \text{if and only if} \quad \underline{\mathbf{a}}^{\alpha} \in T_G$$

On the set $(\mathbf{MO}_k)^n \backslash T_G$, (20) is automatically satisfied, as $f(\underline{\mathbf{b}}) = 0$ for all $\underline{\mathbf{b}} \in (\mathbf{MO}_k)^n \backslash T_G$. Let $\underline{\mathbf{a}} \in T_G$. The coordinates of $\underline{\mathbf{a}}$ lie in exactly p blocks of

$\mathbf{MO}_k$ and any such $\underline{\mathbf{a}}$ is fixed by exactly those automorphisms which only permute atoms in the remaining $k - p$ blocks of $\mathbf{MO}_k$, that is, $|\mathrm{Stab}\ \underline{\mathbf{a}}| = |\mathrm{Aut}(\mathbf{MO}_{k-p})| = 2^{k-p}(k-p)!$, which does not depend on $\underline{\mathbf{a}}$. By (19) the set T_G is partitioned by the automorphism action into orbits of size

$$|\mathrm{Orb}\ a| = \frac{|\mathrm{Aut}(\mathbf{MO}_k)|}{|\mathrm{Stab}\ \underline{\mathbf{a}}|} = \frac{2^k k!}{2^{k-p}(k-p)!} = 2^p \binom{k}{p} p! \tag{21}$$

To define an $\mathrm{Aut}(\mathbf{MO}_k)$-preserving map $f \leq C_G$, we cannot freely choose images from $\mathbf{MO}_k$ for representatives of the orbits within T_G and then use (20) to define the images of the other members of T_G (as we did in the previous paper for $k = 2$), because when $p < k$, there exist automorphisms $\alpha \neq \beta$ such that for any representative $\underline{\mathbf{a}}$ of orbit Orb $\underline{\mathbf{a}}$, $\underline{\mathbf{a}}^\alpha$ is equal to $\underline{\mathbf{a}}^\beta$, restricting the choices for $f(\underline{\mathbf{a}})$ to those which satisfy $f(\underline{\mathbf{a}})^\alpha = f(\underline{\mathbf{a}})^\beta$.

Now for $b \in \mathbf{MO}_k$, $b^\alpha = b^\beta$ if and only if $b^{\alpha\beta^{-1}} = b$ if and only if $\alpha\beta^{-1} \in \mathrm{Stab}\, b$ if and only if $b \in \mathrm{fix}_{\mathbf{MO}_k}(\alpha\beta^{-1})$. In other words, an $\mathrm{Aut}(\mathbf{MO}_k)$-preserving function f is restricted to values $f(\underline{\mathbf{a}}) \in \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$, for $\gamma \in$ Stab $\underline{\mathbf{a}}$, so

$$f(\underline{\mathbf{a}}) \in \bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma) \tag{22}$$

The stabilizer of $\underline{\mathbf{a}}$ consists of exactly those automorphisms which only permute the $k - p$ blocks not covered by the coordinates of $\underline{\mathbf{a}}$ hence $\bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$ is the set of atoms of the p blocks covered by $\underline{\mathbf{a}}$ plus 0 and 1, which are always fixed. When ordered by the usual order relation $\leq$ on $\mathbf{MO}_k$,

$$\bigcap_{\gamma \in \mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma) \cong \mathbf{MO}_p \tag{23}$$

So to construct the $\mathrm{Aut}(\mathbf{MO}_k)$-preserving functions $f\colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$ which are pointwise less than or equal to a given term function $C_G(x_1, \ldots, x_n)$, we need to define f to be zero whenever C_G is and partition the set T_G on which C_G is nonzero into orbits under the automorphism action. By (22) we may freely choose the image $f(\underline{\mathbf{a}})$ for each orbit-representative $\underline{\mathbf{a}}$ within $\bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$, which forces the values of the other points in Orb $\underline{\mathbf{a}}$ to be $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$; so by (23) each orbit within T_G contributes a factor $\mathbf{MO}_p$ to the algebra of $\mathrm{Aut}(\mathbf{MO}_k)$-preserving functions $f\colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$. By (21) the orbits are all of the same size and the number of orbits within T_G is

$$\frac{|T_G|}{|\mathrm{Orb}\ \underline{\mathbf{a}}|} = \frac{2^n \binom{k}{p} p!}{2^p \binom{k}{p} p!} = 2^{n-p}$$

Hence

$$[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}} \tag{24}$$

The interval $[0, c'(x_1, \ldots, x_n)]$ is the product of intervals $[0, C_G(x_1, \ldots, x_n)]$ over all graphs $G = G_p$ $(2 \le p \le k)$ satisfying condition (c) of Proposition 3.2. Now the number of labeled complete p-partite graphs on m vertices is the same as the number of partitions of a labeled m-element set into exactly p parts, which is given by the Stirling numbers $S(m, p)$ of the second kind (Aigner, 1979; 2.66, 3.29, and 3.39):

$$S(m, p) = pS(m-1, p) + S(m-1, p-1) = \frac{1}{p!}\sum_{s=1}^{p}(-1)^{p-s}\binom{p}{s}s^m$$

Since p ranges from 2 to k and the number of isolated vertices l from 0 to $n - p$, the number of the graphs $G = G_p$ on n vertices is given by

$$\phi'(n, p) = \sum_{l=0}^{n-p}\binom{n}{l}S(n-l, p)$$

Note that for $n = 1$, $\phi'(n, p)$ is zero because p is always greater than or equal to 2. Let us define

$$\phi(n, p) = 2^{n-p}\phi'(n, p)$$

It can easily be verified that when $p = 2$, which holds whenever $k = 2$, $\phi(n, 2)$ corresponds to the function $\phi(n)$ in Theorem 1.1, as to be expected. Now by (24) and above, each p, where $2 \le p \le k$, contributes a factor of $(\mathbf{MO}_p)^{\phi(n,p)}$ to the interval $[0, c'(x_1, \ldots, x_n)]$. Hence

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_G [0, C_G(x_1, \ldots, x_n)] \cong \prod_{p=2}^{k} (\mathbf{MO}_p)^{\phi(n,p)}$$

and the final results follow immediately.

Theorem 3.3. For any $n \ge 1$, $k \ge 2$,

$$F_{\mathcal{MO}_k}(n) \cong F_{\mathcal{B}}(n) \times \prod_{p=2}^{k} (\mathbf{MO}_p)^{\phi(n,p)}$$

where $F_B(n)$ is the n-generated free Boolean algebra $\mathbf{2}^{2^n}$,

$$\phi(n, p) = 2^{n-p}\phi'(n, p) = 2^{n-p}\sum_{l=0}^{n-p}\binom{n}{l}s(n-l, p)$$

and the Stirling numbers of the second kind are given by

$$S(m, p) = \frac{1}{p!} \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} s^m$$

Corollary 3.4. For any $n \geq 1$, $k \geq 2$,

$$|F_{\mathcal{MO}_k}(n)| = 2^{2^n} \cdot \prod_{p=2}^{k} (2(p+1))^{2^{n-p}\sum_{l=0}^{n-p}\binom{n}{l}S(n-l,p)}$$

where the Stirling numbers of the second kind are defined by

$$S(m, p) = \frac{1}{p!} \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} s^m$$

Finally, we compute a table from which one can read off the structure of any free algebra $F_{\mathcal{MO}_k}(n)$ for $k, n \leq 10$. If we define $\mathbf{MO}_1$ to be $\mathbf{2}$ and extend the formula $\phi(n, p)$ to include values at $p = 1$ by defining

$$\phi(n, 1) = 2^n$$

then we may write

$$F_{\mathcal{MO}_k}(n) \cong \prod_{p=1}^{k} (\mathbf{MO}_p)^{\phi(n,p)}$$

Hence it is enough to give a table of values of $\phi(n, p)$, for $1 \leq n, p \leq 10$, to describe the structure of $F_{\mathcal{MO}_k}(n)$. To determine these values we need to compute the binomial coefficients $\binom{n}{l}$ and the Stirling numbers of the second kind, $S(n - l, p)$, for $0 \leq l \leq p$. Table I therefore gives part of Pascal's triangle, completed using the recursive definition

$$\binom{n}{0} = 1$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Note that if k is greater than n, then $\binom{n}{k}$ is 0, shown as empty cells in the table.

Table II displays the required Stirling numbers, which can also be defined recursively by

$$S(0, 0) = 1, \qquad S(n, 0) = 0 \quad \text{for } n > 0$$

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

The empty cells are to be filled with 0's again.

Table I. Binomial Coefficients $\binom{n}{k}$ (Pascal's Triangle).

$\binom{n}{k}$	k=0	1	2	3	4	5	6	7	8	9	10
n=1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

The following procedure establishes Table III of values of $\phi(n, k)$: the first column's entries are $\phi(n, 1) = 2^n$. Other entries are given by $\phi(n, k) = 2^{n-k} \sum_{l=0}^{n-k} \binom{n}{l} S(n - l, k)$. For n less than k, $\phi(n, k)$ takes value 0. For $n \geq k$, the sum in the expression is taken over the products of row n entries of Pascal's triangle with column k entries of the Stirling table. The result is then multiplied by 2^{n-k} to give $\phi(n, k)$.

Now, to determine, for example, the structure of the free algebra $F_{\mathcal{MO}_4}$ (6), we consider the first four entries in the 6th row of Table III. The first entry gives the power of $\mathbf{MO}_1 = \mathbf{2}$ in $F_{\mathcal{MO}_4}(6)$, the next one gives the power of $\mathbf{MO}_2$ etc. Thus

$$F_{\mathcal{MO}_4}(6) \cong \mathbf{2}^{64} \times (\mathbf{MO}_2)^{4816} \times (\mathbf{MO}_3)^{2800} \times (\mathbf{MO}_4)^{560}$$

and

$$|F_{\mathcal{MO}_4}(6)| = 2^{64} \cdot (2(2 + 1))^{4816} \cdot (2(3 + 1))^{2800} \cdot (2(4 + 1))^{560}$$

Table II. Stirling Numbers of the Second Kind $S(n, k)$

$S(n, k)$	k=1	2	3	4	5	6	7	8	9	10
n=1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Table III. Values of $\phi(n, k)$

$\phi(n, k)$	$k=1$	2	3	4	5	6	7	8	9	10
$n=1$	2									
2	4	1								
3	8	12	1							
4	16	100	20	1						
5	32	720	260	30	1					
6	64	4816	2800	560	42	1				
7	128	30912	27216	8400	1064	56	1			
8	256	193600	248640	111216	21168	1848	72	1		
9	512	1194240	2182720	1360800	365232	47040	3000	90	1	
10	1024	7296256	18656000	15790720	5743584	1023792	95040	4620	110	1

Also note that, for $k > n$, $F_{\mathcal{MO}_k}(n)$ is equal to $F_{\mathcal{MO}_n}(n)$ and that, for $k < n$, $F_{\mathcal{MO}_{k+1}}(n)$ has an additional nontrivial factor $(\mathbf{MO}_{k+1})^{\phi(n,k+1)}$ when compared to the structure of $\mathbf{F}_{\mathcal{MO}_k}(n)$.

ACKNOWLEDGMENTS

We would like to express our gratitude to H. A. Priestley (Mathematical Institute, Oxford) for her encouragement and helpful remarks. M. H. was partly supported by the Slovak VEGA grant 1/4057/97.

REFERENCES

Aigner, M. (1979), *Combinatorial Theory*, Springer-Verlag, Berlin.

Beran, L. (1984). *Orthomodular Lattices. Algebraic Approach*, Academia, Prague.

Birkhoff, G., and von Neumann, J. (1936). The logic of quantum mechanics, *Annals of Mathematics*, **37**, pp. 823–843 [In *J. von Neumann Collected Works*, Pergamon Press, Oxford (1961), Vol. IV, pp. 105–125.

Bruns, G. (1976). Free ortholattices, *Canadian Journal of Mathematics*, **28**, 977–985.

Clark, D. M., and Davey, B. A. (1996). When is a natural duality "good"? *Algebra Universalis*, **35**, 265–295.

Clark, D. M., and Davey, B. A. (1998). *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge.

Davey, B. A. (1993). Duality theory on ten dollars a day, in *Algebras and Orders*, I. G. Rosenberg and G. Sabidussi, eds., Kluwer Dordrecht, pp. 71–111.

Davey, B. A., and Werner, H. (1983). Dualities and equivalences for varieties of algebras, in *Contributions to Lattice Theory (Szeged, 1980)*, A. P. Huhn and E. T. Schmidt, eds., North-Holland, Amsterdam, pp. 101–275.

Davey, B. A., Haviar, M., and Priestley, H. A. (1995). The syntax and semantics of entailment in duality theory, *Journal of Symbolic Logic*, **60**, 1087–1114.

Grätzer, G. (1978). *General Lattice Theory*, Birkhäuser-Verlag, Basel.

Haviar, M., Konôpka, P., Priestley, H. A., and Wegener, C. B. (1997). Finitely generated free modular ortholattices I, *International Journal of Theoretical Physics*, this issue.

Husimi, K. (1937), Studies on the foundations of the quantum mechanics I., *Proceedings of the Physico-Mathematical Society of Japan*, **19**, 766–789.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London.

MacLaren, M. D. (1964). Atomic orthocomplemented lattices, *Pacific Journal of Mathematics*, **14**, 597–612.

Neumann, P. M., Stoy, G. A., and Thompson, E. C. (1994). *Groups and Geometry*, Oxford University Press, Oxford.

Pták, P., and Pulmanová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.